Indian Statistical Institute
B. Math. Hons. II Year
Semestral Examination 2002-2003
Algebra IV
Duration: 4 Hours

Date:28-04-2003 Marks: 100 Instructor: B. Bagchi

Note: Answer Quesion No.6 and any four questions from the rest. Each question carries 20 marks.

1. Define the character table of a finite group. Show that all the row sums of the table are non-negative integers.

2. (a) If π is a unitary representation of a finite group G on the finite dimensional inner product space V then show that $\frac{1}{|G|} \sum_{x \in G} \pi(x)$ is the orthogonal projection from V onto the subspace V^G of G-fixed vectors.

   (b) If χ_1, χ_2 are two characters of G then show that $\chi_1 \overline{\chi_2}$ is again a character.

   (c) Use the previous parts to deduce that the irreducible characters of G form an orthonormal set in the inner product space $K(G)$ of class functions.

3. If χ is an irreducible character of a finite group G and deg $(\chi) > 1$ then show that $\chi(x) = 0$ for some $x \in G$.
   (Hint: First think of the case in which χ is integer valued.)

4. (a) Show that S_n has exactly two representations of degree 1. What are they?
   (b) Show that any irreducible representation of S_n of degree ¿ 1 has degree $\geq n - 1$.
   (c) Show that S_n has exactly two irreducible representations of degree $n - 1$ and describe them explicitly.
   (d) If H is a proper subgroup of S_n then show that either $H = A_n$ or the index of H is at least n.
   (Hint: You may recall the branching theorem.)

5. State the determinantal formula for the irreducible characters of S_n, with complete explanations. Deduce the hook length formula for the degrees of these characters.

6. Use the determinantal formula to compute the character table of S_5.